On black holes as inner boundaries for the constraint equations

S. Dain
Max-Planck-Institut für Gravitationsphysik
Am Mühlenberg 1
14476 Golm
Germany
e-mail: dain@aei.mpg.de

February 7, 2008

Abstract

General aspects of the boundary value problem for the constraint equations and their application to black holes are discussed.

1 Introduction

There are many kinds of boundaries in physics. For example, in electrodynamics, the boundary given by the interface between a charge distribution and vacuum. However, this is not a fundamental boundary of Maxwell equations, in the following sense. This boundary is introduced in the sources by choosing a charge density with compact support. The sources satisfy extra equations. Maxwell equations are expected to be fundamental equations; matter sources equations are phenomenological approximation suitable to describe some specific matter models. The same apply to matter sources in Einstein equations. In this case the space time boundary is introduced in the sources of Einstein equations by choosing an energy density with compact support.

In the case of vacuum Maxwell equations, the field does not interact with it self, hence it can not produce "its own boundary". In the case of Einstein vacuum equation there exists such fundamental kind of boundary produced only by the self interaction of the vacuum field: black holes.

In the context of an initial value formulation, the first step in order to understand the space time boundary produced by a black hole is the study of the intersection of this boundary with a spacelike three-dimensional Cauchy hypersurface. That is, to study the black hole boundary value problem for the constraint equations. This problem has been recently studied in [11] [9].

Since only fundamental properties of gravity are involved in a vacuum black hole, it can be expected that black hole boundary conditions can be

written in a geometrical form. It turns out that this is true. Moreover, black holes boundaries for the constraint equations suggest a deep interplay between Riemannian geometry, elliptic equations and physics. In the present article we discuss some general aspects about this interplay.

In section 2 we present the constraint equations and the corresponding boundary conditions. In section 3 we discuss elliptic reductions to this equation. General kind of boundary conditions are discussed in section 4. In section 5 we discuss boundary conditions that are physically meaningful. Finally, in section 6, we discuss black hole boundary conditions.

2 The constraint equations

Let C_k be a finite collection of *compact* sets in $\mathbb{R}^3$. We define the *exterior region* $\tilde{\Omega} = \mathbb{R}^3 \setminus \cup_k C_k$. For simplicity, we will mainly consider the constraint equations in the time-symmetric case, that is, when extrinsic curvature vanishes. However, all the following consideration apply also to the general case (see [9]). Let $\tilde{h}_{ab}$ be a Riemannian metric on $\tilde{\Omega}$ and let $\tilde{R}$ be the corresponding Ricci scalar. The time-symmetric, vacuum, constraint equation is given by

$$\tilde{R} = 0, \tag{1}$$

on $\tilde{\Omega}$.

There exists two kinds of boundary conditions for equation (1) in $\tilde{\Omega}$: outer boundary conditions and inner boundary conditions. The outer boundary condition is asymptotic flatness, and it is essentially a fall of condition on $\tilde{h}_{ab}$. Physically, it means that we have an isolated system. The data will be called *asymptotically flat* if there exists some compact set C, with $\cup_k C_k \subset C$, such that $\tilde{\Omega} \setminus C$ can be mapped by a coordinate system $\tilde{x}^j$ diffeomorphically onto the complement of a closed ball in $\mathbb{R}^3$ and we have in these coordinates

$$\tilde{h}_{ij} = (1 + \frac{2m}{\tilde{r}})\delta_{ij} + O(\tilde{r}^{-2}), \tag{2}$$

as $\tilde{r} = (\sum_{j=1}^{3}(\tilde{x}^j)^2)^{1/2} \to \infty$, where the constant m is the total mass of the initial data.

The inner boundary condition will be the black hole boundary condition. The boundaries ∂C_k are assumed to be smooth, two dimensional surfaces in $(\tilde{\Omega}, \tilde{h})$. Let $\tilde{\nu}^a$ be the unit normal of ∂C_k, with respect to $\tilde{h}_{ab}$, pointing in the *outward* direction of $\tilde{\Omega}$. Let t^a be the unit timelike vector field orthogonal to the hypersurface $\tilde{\Omega}$ with respect to the spacetime metric g_{ab} ($t^a t^b g_{ab} = -1$ with our signature convention) The outgoing and ingoing null geodesics orthogonal to ∂C_k are given by $l^a = t^a - \tilde{\nu}^a$ and $k^a = t^a + \tilde{\nu}^a$ respectively, the corresponding expansions are given by $\Theta_+ = \nabla_a l^a$ and $\Theta_- = \nabla_a k^a$, where ∇_a is the connexion with respect to g_{ab}. We can calculate these expansions in terms of quantities intrinsic to the initial data. In the particular case of time-symmetric data we have

$$\Theta_- = -\Theta_+ = \tilde{H}, \tag{3}$$

where $\tilde{H} = \tilde{D}_a \tilde{\nu}^a$, and $\tilde{D}_a$ is the covariant derivative with respect $\tilde{h}_{ab}$. That is, $\tilde{H}$ is the mean curvature of the two-dimensional surface $\partial\tilde{\Omega}$ with respect to the metric $\tilde{h}_{ab}$ and the normal $\tilde{\nu}^a$. The boundary will be called *marginally trapped* if

$$\tilde{H} = 0. \tag{4}$$

A marginally trapped surface indicates the presence of a black hole (see [14] and also the discussion in [9]). Equation (4) geometrically means that the two dimensional boundary is an extremal surface with respect to the metric $\tilde{h}_{ab}$. Equation (4) will be our inner boundary condition.

3 The constraint equations as an elliptic system

We want to find solutions $\tilde{h}_{ab}$ to equation (1) which satisfy boundary conditions (2) and (4). If we write equation (1) in terms of the metric components of $\tilde{h}_{ab}$ we get a complicated non linear equation. We have six unknown functions in the metric $\tilde{h}_{ab}$ and only one equation. That is, we have an underdetermined system. The strategy to solve this equation is to split the six unknowns into two sets. One set will be called the *free data*, we want to prescribe them freely or at least with some restrictions easy to achieve, which, in particular, do not involve solving differential equations. Roughly speaking, the free data set should contain five free functions. The other set will contain only one function. For a given choice of free data we have to solve equation (1) to calculate this function.

There exists a priori many ways of doing this splitting. For example we can chose some coordinate system and chose the free data set to be five components of the metric in these coordinates and try to solve the equation for the remainder. But the resulting equation will be in general a complicated non linear equation which is of no known type and hence will be difficult, if not impossible, to prove that in fact for every choice of free data we do get a solution for the remainder.

In order to control the behavior of the solution $\tilde{h}_{ab}$ at the boundary we need also to prescribe boundary conditions for the unknown function. Our splitting and boundary conditions will be successful if for arbitrary free data and boundary conditions we always get a solution for the reminder. This suggests that the problem has an elliptic nature. To some extend this is true, the constraint equation can be reduced to an elliptic system. However, this is not the only way of solving them, for example they can be reduced to a parabolic system [3] and this lead to the discovery of new kinds of solutions. Nevertheless, so far only the elliptic approach has been successful in the general case (the parabolic system has been mainly studied in the time symmetric case).

There is not a unique way of getting an elliptic system out of equation (1). Different elliptic systems will lead to different choices of free data and boundary conditions, and hence they can be more appropriate to describe different kinds of physical situations. Here we will use the so called conformal method, which is probably the simplest one, since it lead to semilinear equations, which reduce to a linear equation in the time-

symmetric case. Other elliptic reductions give quasi linear equations [7] [8] [6] [5][4]. So far, black hole boundary conditions have been only studied using the conformal method [11] [9]. It will very be interesting to study them with other elliptic reductions.

The conformal method is as follows. Let h_{ab} be a Riemannian metric, let ψ be a *positive* solution of the following equation

$$L_h\psi = 0, \tag{5}$$

where $L_h \equiv D^a D_a - R/8$, D_a is the covariant derivative with respect to h_{ab} and R is the Ricci scalar of the metric h_{ab}. Then, the rescaled metric $\tilde{h}_{ab} = \psi^4 h_{ab}$ will satisfy equation (1). Note that the differential operator L_h is elliptic, hence the linear equation (1) is an elliptic reduction to the time symmetric constraint equation (1).

Two metrics $\hat{h}_{ab}$ and h_{ab} belong to the same conformal class if there exists a positive conformal factor $\hat{\psi}$ such that $\hat{h}_{ab} = \hat{\psi}^4 h_{ab}$. For any metric in the same conformal class we get the same solution $\tilde{h}_{ab}$. That is, the free data set is given by the conformal class of metrics, which can be represented by five free functions.

4 Elliptic boundary conditions

What kind of boundary conditions are compatible with the constraint equations? If we have reduced them to an elliptic system, this question can be answer in full generality. Given an elliptic system, the boundary conditions will lead to a well posed problem if and only if they satisfy the so called *complementing condition* or *Lopatinski-Schapiro* conditions (see [1] [2] for a precise statement and also the introductory book [13]). These are conditions at the linear level. For non linear system, these conditions are imposed to the associated linearized problem. Of course in the non linear case, these conditions are in general only necessary but not sufficient to prove the existence of solution. For non linear systems we have to study each particular case to decide whether there exists solutions or not.

Simple examples of boundary conditions that satisfy the Lopatinski-Schapiro requirements for equation (1) are Dirichlet and Neumann boundary conditions for the conformal factor ψ. More general boundary conditions are possible, it is even possible that the order of the derivatives in the boundary operator is higher than the order of the derivatives in the differential operator.

However, here we are interested in positive solutions. This is an extra requirement of our particular elliptic reduction. If ψ is zero at some point then $\tilde{h}_{ab} = \psi^4 h_{ab}$ will not be a Riemannian metric at this point. The positivity of the solution can be proved by a particular feature of second order elliptic equations: the maximum principle (see, for example, [10]).

In order to use the maximum principle we need an extra requirement on the lower order coefficients of the operator L_h

$$R \geq 0. \tag{6}$$

The most general kind of boundary condition that satisfy the Lopatinski-Shapiro conditions and also allow us to use the maximum principle to prove positivity is given by the Dirichlet boundary condition

$$\psi = \varphi \quad \text{on } \partial\tilde{\Omega} \tag{7}$$

or the oblique derivative boundary condition

$$\beta^a D_a \psi + \alpha\psi = \varphi \quad \text{on } \partial\tilde{\Omega}, \tag{8}$$

where φ, α and β^a are arbitrary functions which satisfy $\varphi \geq 0$, $\alpha \geq 0$ and $\beta^a \nu_a > 0$ on $\partial\tilde{\Omega}$, where ν^a is the outward unit normal with respect to the conformal metric h_{ab}. We note that $\beta^a \nu_a > 0$ guarantee that (8) satisfies the Lopatinski-Schapiro conditions.

In the exterior region $\tilde{\Omega}$ we need in addition the asymptotic flatness condition which in this case is given by

$$\lim_{r\to\infty} \psi = 1. \tag{9}$$

Conditions (6), (7) or (8), (9) will guarantee the existence of a unique positive solution ψ of equation (5), and hence an asymptotically flat metric $\tilde{h}_{ab}$ which satisfies the constraint equation (1) with (7) or (8) as inner boundary condition. Of course, for arbitrary φ, α and β^a, the boundary condition (8) will not have any interesting geometrical meaning.

5 On physical boundary conditions

Probably any smooth solution of the constraint equation in some bounded region can have some physical interpretation in the sense that this bounded region can be a piece of a space time that can describe some physical phenomena. The situation is different in the case of an exterior region $\tilde{\Omega}$: in order to be physically meaningful as a description of an isolated system the solution must have positive mass. Since we artificially cut out a region in $\mathbb{R}^3$, the positivity mass theorem does not automatically apply. Many of the solutions found in the previous section will have negative total mass. This can be explicitly seen in the following example.

Let us consider the Schwarzschild, time symmetric, initial data. In this case $h_{ab} = \delta_{ab}$, where δ_{ab} is the flat metric, $L_h = \Delta$, where Δ is the flat Laplacian and the conformal factor is given by

$$\psi = 1 + \frac{m}{2r}. \tag{10}$$

We chose the exterior region $\tilde{\Omega}$ to be the exterior of a ball of radius $r = a$. Note that with our conventions $\nu^a = -(\partial/\partial r)^a$. The Dirichlet boundary condition is given by

$$\psi = \varphi_0 \quad \text{on } \partial\tilde{\Omega}, \tag{11}$$

where φ_0 is a positive constant. Using (10) one easily check that $\varphi_0 < 1$ implies $m < 0$ and $\varphi_0 > 1$ implies $m > 0$. Take $a > m/2$, then we will have $\psi > 0$ in the exterior region and $m < 0$ if we chose $\varphi_0 < 1$. This means that $\varphi_0 < 1$ is not a physical boundary condition although it mathematically consistent in the sense that it gives us existence and uniqueness

of a positive solution. In the general case, if we impose Dirichlet boundary conditions to the conformal factor appears to be difficult to recognize for which value of the boundary function we will get data with positive mass. We conclude that Dirichlet boundary conditions for ψ are not physically meaningful in general.

In this example, the Neumann condition

$$\nu^a D_a \psi = -\left.\frac{\partial \psi}{\partial r}\right|_{r=a} = \varphi_0 \tag{12}$$

with $\varphi_0 > 0$ always produce data with positive mass, since $m = 2\varphi_0 a^2$. This can be generalized. Consider the case of non trivial extrinsic curvature, and assume that the trace of it is zero. Assume that the conformal metric h_{ab} satisfies $R = 0$ and the following fall off $h_{ij} = \delta_{ij} + O(r^{-2})$. The equation for the conformal factor is given by

$$L_h \psi = D^a D_a \psi = -\frac{K_{ab} K^{ab}}{8\psi^7}, \tag{13}$$

where K_{ab} is the rescaled extrinsic curvature. Let us impose Neumann boundary condition to this equation

$$\nu^a D_a \psi = \varphi \quad \text{ on } \partial\tilde{\Omega}, \tag{14}$$

with $\varphi \geq 0$. Integrating equation (13) on $\tilde{\Omega}$ and using the Gauss theorem we get

$$m \geq \int_{\partial\tilde{\Omega}} \nu^a D_a \psi \, dS \geq 0, \tag{15}$$

where we have used (14) and the following expression for the mass

$$m = -\frac{1}{4\pi} \lim_{r\to\infty} \int_{S_r} \frac{\partial \psi}{\partial r} \, dS_r. \tag{16}$$

We conclude that every data which satisfy the boundary condition (14) in the appropriate conformal class will have positive mass. Note that the boundary condition (14) is linear even in the general, non time-symmetric, case. It is not clear to me the meaning of this condition, and I am not aware of any application of it. Although in [11] only black hole boundary conditions has been studied, remarkably, all the solutions founded there satisfy in addition condition (14).

6 Black hole boundary conditions

The most important inner boundary condition for the vacuum constraint equation is the black hole condition (4). This boundary condition has both a geometric and physical meaning. There exist versions of the positivity mass theorem which include black holes as inner boundaries (see for example [12]).

Condition (4) can be written in terms of the conformal quantities as

$$4\nu^a D_a \psi + H\psi = 0, \tag{17}$$

where $H = D_a \nu^a$.

If $H \geq 0$ on $\partial\tilde{\Omega}$, then condition (17) has the form (8). If, in addition, the conformal metric h_{ab} satisfies $R \geq 0$ on $\tilde{\Omega}$ we can apply the standard linear elliptic theory and the maximum principle to prove that there will exist a unique positive solution ϕ of our problem.

The conformal metric is not really a free data, it should satisfy the condition $R \geq 0$, and also the boundary $\partial\tilde{\Omega}$ should satisfy $H \geq 0$. Note that if we chose the conformal metric to be the flat metric and $\partial\tilde{\Omega}$ any sphere, then this boundary will have $H < 0$, and hence it does not satisfy our hypothesis. However, it is simple to construct families of metric which satisfy $R \geq 0$ and $H \geq 0$ on $\partial\tilde{\Omega}$. Consider, for example, the time-symmetric initial data for Reissner-Nordstrom. The metric is given by

$$h_{ab} = \hat{\psi}^4 \delta_{ab}, \tag{18}$$

where

$$\hat{\psi} = \frac{1}{2r}\sqrt{(q + 2r + m)(-q + 2r + m)}. \tag{19}$$

The constants q and m are the charge and the mass of the data respectively. We assume $q^2 < m^2$. The Ricci scalar is given by

$$R = \frac{2q^2}{\hat{\psi}^8 r^4}, \tag{20}$$

which is positive. For $r < r_0 = (\sqrt{m^2 - q^2})/2$ the mean curvature of the two surfaces of constant radius is positive. Then this metric satisfies our hypothesis. Moreover, take $q > 0$ (this implies $R > 0$), let $\partial\tilde{\Omega}$ such that $H > 0$, let ϵ small enough and λ_{ab} and arbitrary tensor field; then the metric $h_{ab} + \epsilon\lambda_{ab}$ satisfies also $R > 0$ and $\partial\tilde{\Omega}$ will satisfy $H > 0$ with respect to this metric.

In the general case, black hole boundary conditions are non linear. This introduces extra difficulties in both the existence proof and the election of free data. This problem has been recently studied in [11] [9]. In those references, large classes of black hole exterior regions have been constructed. However, it is still an open problem how to construct and characterize *all* possible initial data for black holes exterior regions.

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